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Frontogenesis in a three-dimensional linear velocity field

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FRONTOGENESIS IN A THREE-DIMENSIONAL LINEAR VELOCITY FIELD

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Department of Aerology.

Approved:

Academic Dean.

PREFACE

The investigation of frontogenesis in a three-dimensional linear velocity field was commenced in the fall of 1947 at the U. S. Naval Postgraduate School in partial fulfillment of the requirements for the degree of Master of Science.

The two-dimensional development of frontogenesis by Petterssen (5,6) aroused the desire to visualize and to attempt to portray what would occur in an extension of his definition of frontogenesis into three dimensions. This paper covers the main essentials in the development. Time limitations have prevented a more exhaustive investigation.

The author wishes to express his appreciation for general guidance to Professor W. D. Duthie, and for assistance by the staffs of the departments of Aerological Engineering and Mathematics of the U. S. Naval Postgraduate School.

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TABLE OF SYMBOLS AND ABBREVIATIONS

$\underline{V}$	vector velocity
i,j,k	unit vectors along the x,y,z axes, respectively, in a right handed coordinate system
$\underline{r}$	position vector
x,y,z	rectangular coordinates
u,v,w	scalar components of vector velocity, $\underline{V}$, along the x,y,z axes, respectively
u_o, v_o, w_o	constant velocity components at the point (Ø,Ø,Ø)
$u_x,(u_y)$, etc.	partial differentiation of u with respect to x,(y), etc.
$\Phi,(\Phi_c)$	tensor, (conjugate of the tensor)
$\omega_x,(\omega_y)$, etc.	angular velocity of rotation about the x,(y), etc. axis
$K_1,(K_2)$, etc.	arbitrary constant arising from the process of integration
λ,μ,ν	arbitrary constant multipliers
ρ_1, ρ_2, ρ_3	roots (real and/or complex) of a cubic equation
$\Omega_1, \Omega_2, \Omega_3$	linear functions of x,y,z
$a+bi$, etc.	complex number of real part, a, and imaginary part, b
$\ln a$	natural logarithm of a (to the base e)
l,m,n	direction cosines
f	frontogenetical function
S	any conservative property that may be observed in the free atmosphere

$\lvert \nabla S \rvert$	absolute magnitude of the gradient of property.
Fg	frontogenesis.
Fl	frontolysis.

I. INTRODUCTION

The first available source of information on frontogenesis is by Bergeron (1). He gives a limited treatment of the deformation field only, in three dimensions. Bjerknes (2) later treats the case of the deformation field in two dimensions. The first full treatment of frontogenesis is given by Petterssen (5,6). His treatment is limited to two dimensions, however.

This paper is a generalization of the two-dimensional theory of frontogenesis in a linear velocity field as developed by Petterssen (5).

The most general homogeneous linear velocity field in three dimensions contains nine constants to describe the motions of translation, divergence, deformation, and rotation. The two-dimensional specialization of the general case contains five constants to describe the same motions.

An investigation of the possible frontogenetical and frontolytical sectors determined by the surface for which the frontogenetical function, f, is equal to zero yields the following:

(1) The surface for which f is equal to zero, if it exists, is an elliptic cone with vertex at the origin of the coordinate system.

(2) The wind field, as described by the nine constants mentioned above, determines whether or not the surface for

which f equals zero will exist. If it does exist, certain of these constants determine just what the shape, size, and orientation of the conical sectors will be.

(3) If the direction of the gradient of property being considered lies (a) within the frontogenetical sector, (b) on the surface for which f equals zero, or (c) outside the frontogenetical sector, there will take place (a) frontogenesis, (b) no change, or (c) frontolysis, respectively.

The conservative property used in this paper is potential temperature whose equipotential surfaces are nearly horizontal and whose gradient is nearly vertical. Instability can cause the gradient of property to depart from the vertical.

An application of the theory of frontogenesis is made in each of the following types of wind systems observed in the free atmosphere: cols, cyclones, anticyclones, troughs, and wedges. In each of these cases only the normal situation is treated in this paper. It is found that in the case of a col with convection at its center frontolysis will occur at the center, while in a col with subsidence at its center frontogenesis will occur at the center. In the case of a cyclone with convection at its center and attendant inflow in the horizontal plane frontolysis will take place at the center. In an anticyclone with subsidence at its center and attendant outflow in the horizontal plane frontogenesis will take place at the center. Troughs and wedges in this paper are considered as special cases of cyclones and anticyclones, respectively. Frontogenesis or frontolysis occurs as is indicated for the cyclones or anticyclones, if the direction of the gradient of property is assumed nearly vertical.

Variations in the frontogenetical effect can occur in the above systems in the area around the centers of the systems where marked convection or subsidence can cause the direction of the gradient of property to depart from the vertical.

II. LINEAR FIELDS OF MOTION.

1. Linear Vector Function of Position.

For the velocity, $\underline{V}$, the linear vector function of position can be written as

(1-1) $$\underline{V} = \underline{V_0} + \underline{r} \cdot \nabla \underline{V}.$$

In Cartesian coordinates this becomes

(1-2)
$$\begin{aligned} u &= u_0 + x\, u_x + y\, u_y + z\, u_z \\ v &= v_0 + x\, v_x + y\, v_y + z\, v_z \\ w &= w_0 + x\, w_x + y\, w_y + z\, w_z . \end{aligned}$$

shl [illegible]

This development follows that of Prandtl and Tietjens (7) and is given in section 1 of Appendix I.

The equations above define the velocity field in the vicinity of a point where the second order and higher terms of the Taylor series are negligible as compared to the linear terms.

2. The Equations of Linear Fields of Motion.

The equations (1-2) are transformed in section 2 of Appendix I so that the equations of linear fields of motion are

(2-1)
$$\begin{aligned} u &= u_0 + Ax + Dy + Ez \\ v &= v_0 + By - Dx + Fz \\ w &= w_0 + Cz - Ex - Fy , \end{aligned}$$

or, rewriting,

(2-2)
$$\begin{aligned} u &= u_0 + (A+B+C)x - (B+C)x + Dy + Ez \\ v &= v_0 + (A+B+C)y - (A+C)y - Dx + Fz \\ w &= w_0 + (A+B+C)z - (A+B)z - Ex - Fy , \end{aligned}$$

where: u_0, v_0, w_0 are the constants of translation; $(A+B+C)$ is the constant of divergence; $(B+C)$, $(A+C)$, $(A+B)$ are the constants of deformation; D, E, F are the constants of rotation; and A, B, C are the constants of extension along the x, y, z axes, respectively.

3. Streamlines.

A streamline is a line for which the tangent at any point gives the direction of the fluid motion at that point.

The equations of section 2 will be discussed in terms of the streamlines for convenience. The differential equations of the streamlines are

$$(3\text{-}1) \qquad \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad .$$

These equations will be used to study, first, the separate components of translation, divergence, deformation, and rotation, and then the general case in which all of the components are included.

4. Translation Field.

Considering only the component of translation of equations (2-2), solutions of the differential equations give

$$(4\text{-}1) \qquad \frac{x-K_1}{u_0} = \frac{y-K_2}{v_0} = \frac{z-K_3}{w_0}$$

where u_0, v_0, w_0 are the direction numbers of a family of straight, parallel streamlines through the points determined by the arbitrary constants K_1, K_2, K_3. This is a rectilinear field of motion.

5. Divergence Field.

Considering the component of divergence only, solutions of the differential equations give

(5-1) $$K_4 x = K_5 y = K_6 z \, .$$

The streamlines are straight, radial lines emanating from the origin of the coordinate system. Mathematically, the origin is a source of fluid if the divergence is positive, and it is a sink for the fluid if the divergence is negative (convergence). These cases are (A+B+C) greater than, or less than, zero, respectively.

In this paper, divergence refers to mass divergence which can express itself only as a change in the fluid density. In the two-dimensional case as presented by Stewart (8), the divergence is a "velocity" divergence where the third dimension acts as the source or sink of fluid, and where the mass divergence is equal to zero. Such a velocity divergence is represented as the three-dimensional deformation field.

6. Deformation Field.

If only the component of deformation is considered, solutions of the differential equations are

(6-1) $$xyz = K_7 \qquad x^{+1/A'} = K_8\, y^{-1/B'}$$
$$x^{+1/A'} = K_9\, z^{\pm 1/C'}$$

where
$$-(B+C) = A$$
$$-(A+C) = B$$
$$-(A+B) = C$$

since A+B+C equals zero. Choose axes so that A is always positive, B is always negative, and C is either positive or negative. Let this sign convention be denoted as

$$(6\text{-}3)\qquad \begin{aligned} A &\equiv +A' \\ B &\equiv -B' \\ C &\equiv \pm C' \,. \end{aligned}$$

This convention causes the x axis to be an axis of dilatation, the y axis to be an axis of contraction, and the z axis to be an axis of dilatation if C' is positive and an axis of contraction if C' is negative. When A' and C' are positive with B' negative, this causes the xz plane to be a plane of outflow and the y axis an axis of inflow. When A' is positive with B' and C' negative, this causes the yz plane to be a plane of inflow and the x axis an axis of outflow. In applying the above solutions, or the general solutions that follow, to actual examples, it is necessary to follow this sign convention established in order that succeeding developments will apply.

From the above solutions it is seen that two streamline patterns can be obtained depending on the sign of C'. If C' is positive (negative) the streamline pattern will be similar to that shown in figure 1 (2). Positive (negative) C' represents outflow (inflow) along the z axis, with outflow along the x axis and inflow along the y axis. For C' positive (negative), this is the deformation a sphere would undergo in changing into an oblate (prolate) spheroid. Refer to figure 3.

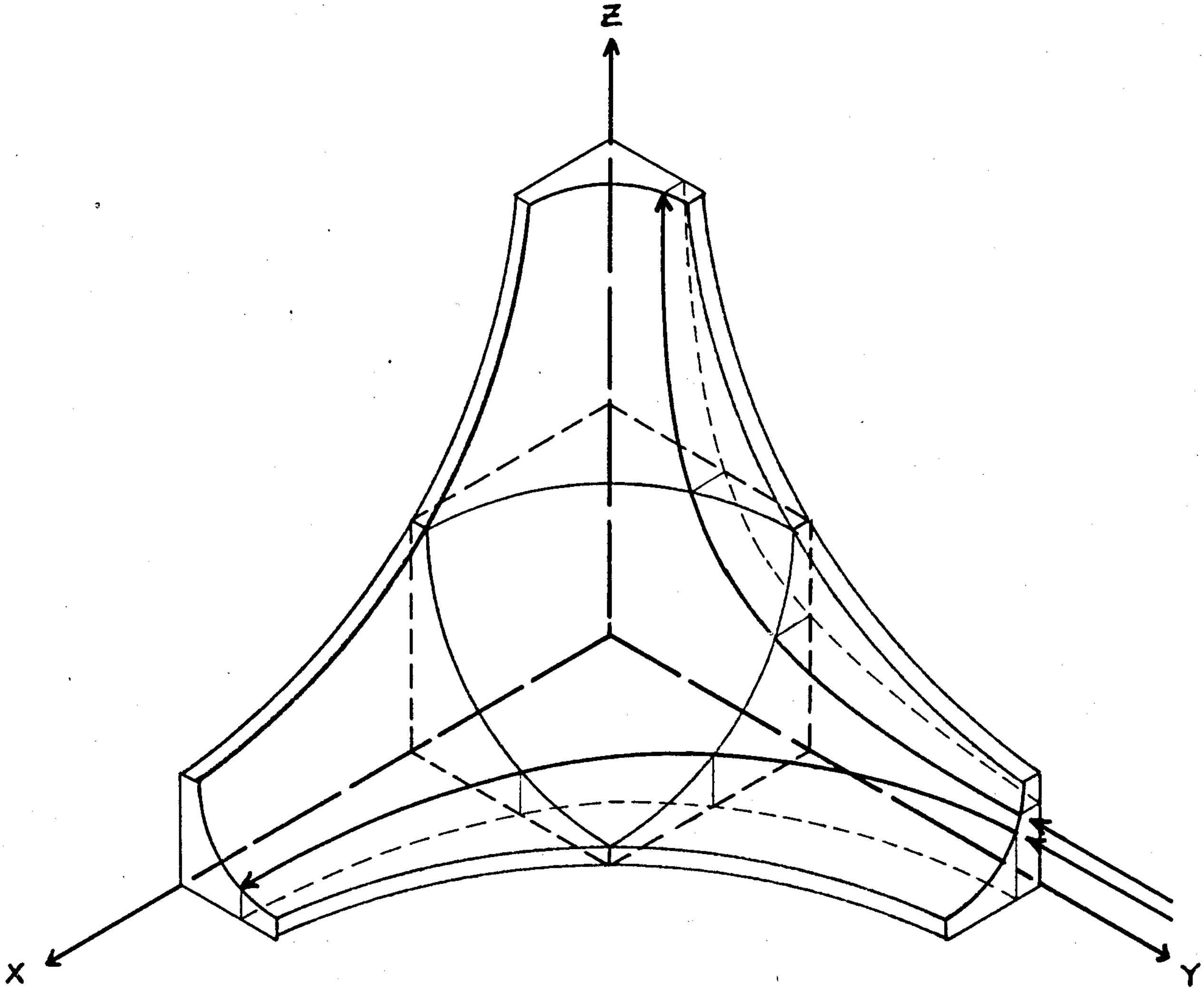

Figure 1.

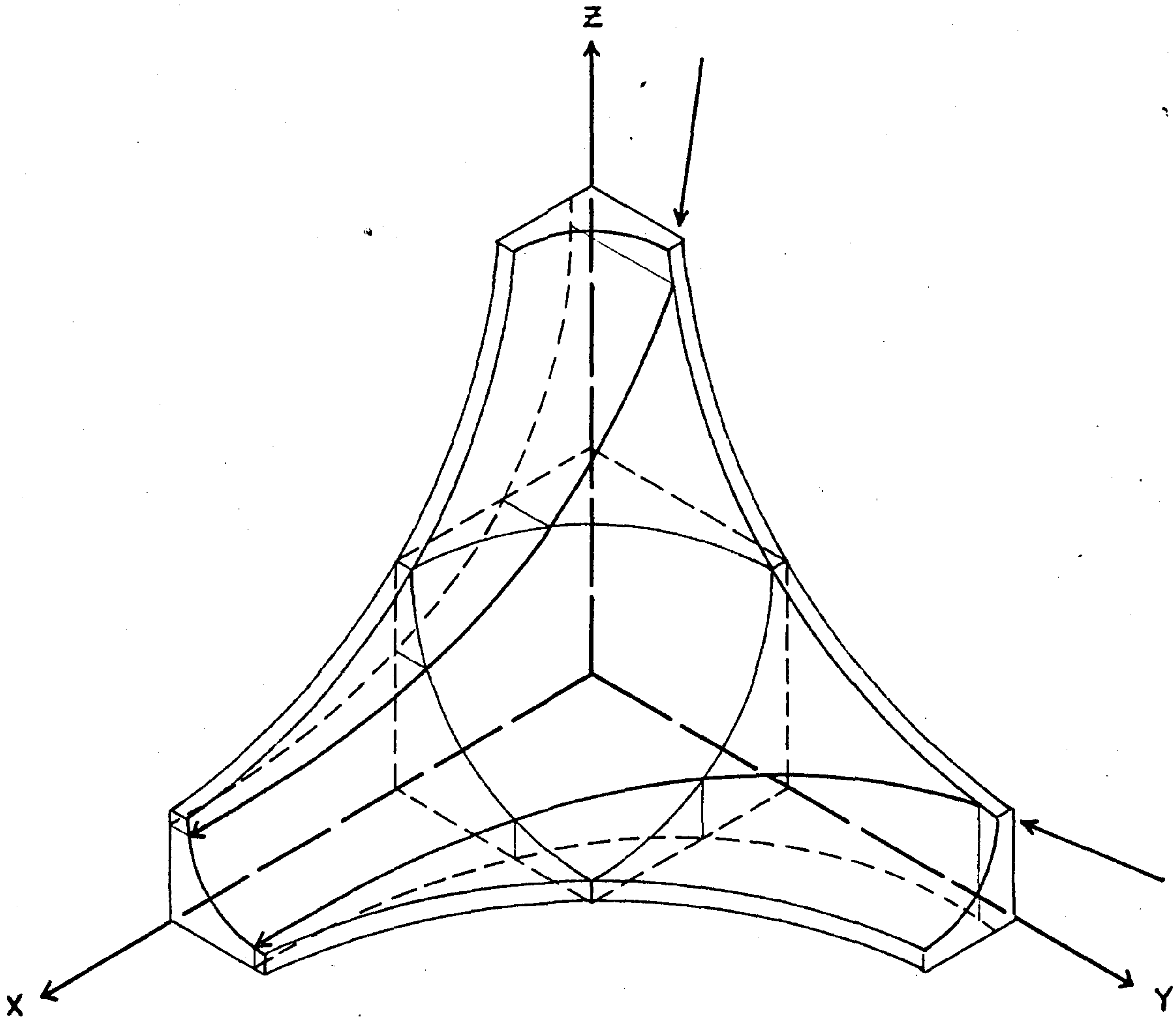

Figure 2.

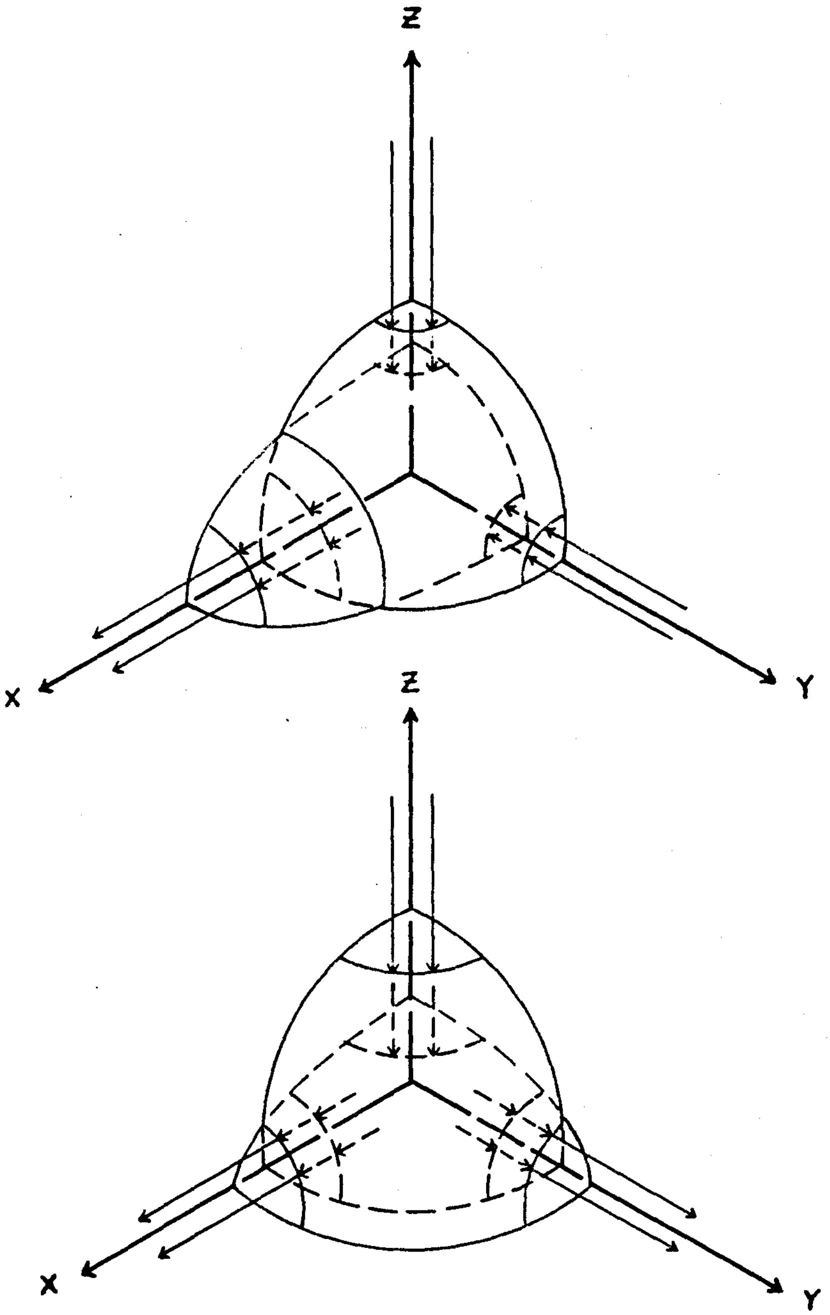

Figure 3.

7. Rotation Field.

Considering only the component of rotation, solutions of the differential equations are

$$(7\text{-}1) \qquad x^2 + y^2 + z^2 = K_{10}^2 \qquad Fx - Ey + Dz = K_{11}$$

where K_{10} is greater than K_{11} for streamlines to exist. A simultaneous solution of these equations gives a family of circular streamlines all with a common axis of rotation defined by the direction numbers F,-E,D.

8. The General Field.

In order to solve the differential equations when all the components, except translation, are considered it is necessary to make the equations exact. There exist multipliers such that the numerators are constant multiples of the differentials of their respective denominators. These multiples are designated ρ_1, ρ_2, ρ_3. They are the roots of a cubic equation, and as such there may be all real roots or one real root and two conjugate complex roots. This derivation is due to Ford (3) and will be found in Appendix II.

For the case of all real roots the solution is

$$(8\text{-}1) \qquad \Omega_1^{1/\rho_1} = K_{12}\,\Omega_2^{1/\rho_2} = K_{13}\,\Omega_3^{1/\rho_3}$$

where $\Omega_1, \Omega_2, \Omega_3$ are linear functions of x,y,z with real terms only. A simultaneous solution of these equations gives skew space curves for the streamlines similar to those shown in figure 4. The sign convention of section 6 applies here to the oblique planes and axes. For A'

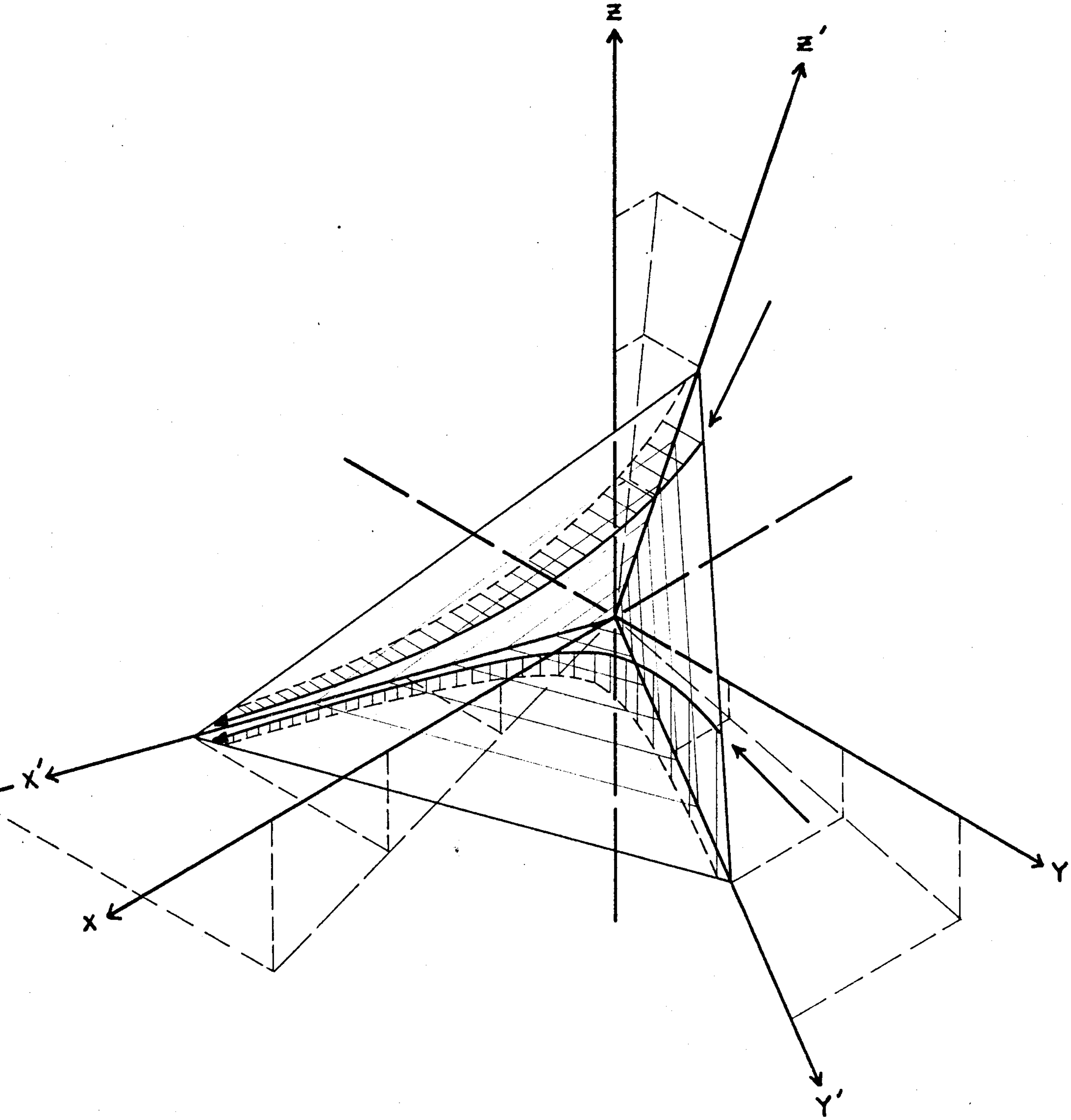

FIGURE 4.

and C' positive with B' negative, the xz plane is a plane of dilatation; the x'z' plane is a plane of outflow; the y axis is an axis of contraction; and the y' axis is an axis of inflow. For A' positive with B' and C' negative, the yz plane is a plane of contraction; the y'z' plane is a plane of inflow; the x axis is an axis of dilatation; and the x' axis is an axis of outflow.

For the case of imaginary roots, it is shown in the Appendix that the solution is of the form

$$
\begin{aligned}
\tfrac{a}{\rho_1}\ln \Omega_1 &= \ln K_{14} + \tfrac{1}{2}\ln\left[(cx+ey+gz)^2 + (dx+fy+hz)^2\right] \\
\tfrac{b}{\rho_1}\ln \Omega_1 &= \arctan \frac{dx+fy+hz}{cx+ey+gz}
\end{aligned}
\qquad (8\text{-}2)
$$

where Ω_1, is a linear function of x,y,z with real terms only. A simultaneous solution of these equations gives skew space curves of a general helical, or spiral, shape. Refer to figure 5.

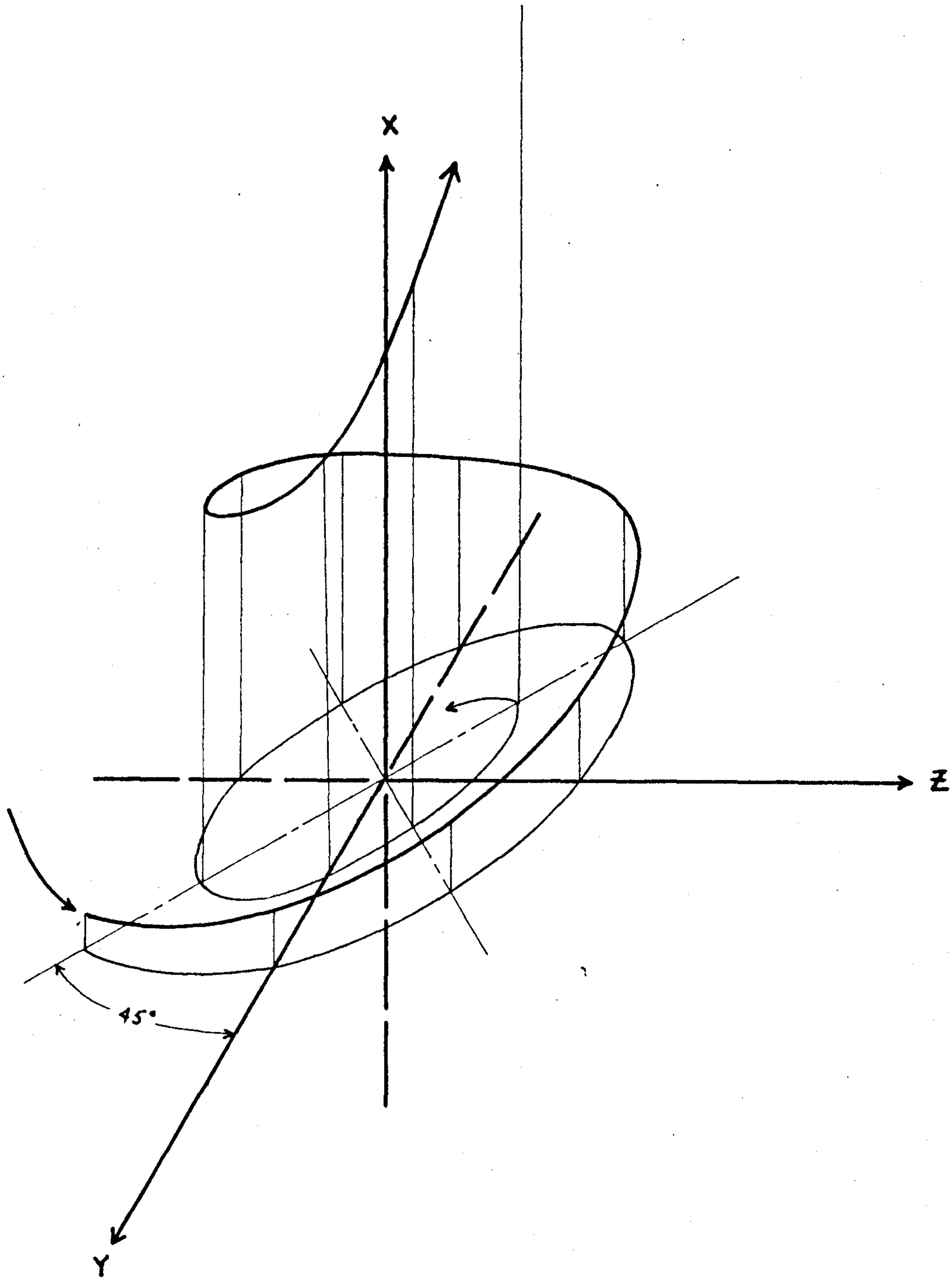

FIGURE 5.

III. FRONTOGENESIS IN GENERAL.

9. Definition of Frontogenesis.

The definition of frontogenesis as given by Petterssen (5) will be used. Essentially, it is the time rate of change of the gradient of a conservative property which can be written

(9-1) $$f = \frac{d}{dt} |\nabla S| \; .$$

Petterssen has developed this definition further, so that it becomes

(9-2) $$f = -\nabla S \cdot (\nabla S \cdot \nabla) \underline{V} \Big/ |\nabla S| \; .$$

10. Frontogenesis in Linear Fields of Motion.

When equation (9-2) is applied to the equations for linear fields of motion (2-1), the following equations result:

(10-1) $$f = -|\nabla S| \left[A\,\ell^2 + B\,m^2 + C\,n^2 \right]$$

(10-2) $$f = |\nabla S| \left[(B+C)\,\ell^2 + (A+C)\,m^2 + (A+B)\,n^2 - (A+B+C) \right]$$

where l,m,n are the direction cosines of the gradient of property. The intermediate steps in the derivation are given in Appendix III. These equations show that the frontogenetical effect is independent of both translation and rotation. Divergence deducts from the frontogenetical effect; that is, it produces frontolysis. Convergence adds to the frontogenetical effect; produces frontogenesis. The bearing deformation has on the frontogenetical effect depends on the values of the deformation constants. It is obvious that if there were no velocity of extension occurring in the atmospheric wind field, there would be no frontogenetical effect.

The surface in which the absolute magnitude of the gradient of property is a maximum coincides with the surface of frontogenesis.

11. Frontogenetical and Frontolytical Sectors.

The equations (10-1) and (10-2) show that the sign of f is determined by the quantity within the brackets. An investigation of the surface for which the frontogenetical function is equal to zero yields a description of the frontogenetical (f greater than zero) and frontolytical (f less than zero) sectors with relation to the gradient of property.

For f equal to zero,

(11-1) $$A\ell^2 + Bm^2 + Cn^2 = 0 ,$$

or

(11-2) $$(B+C)\ell^2 + (A+C)m^2 + (A+B)n^2 - (A+B+C) = 0 .$$

These equations represent an elliptic cone with vertex at the origin of the coordinate system. From the sign convention established in equation (6-3), two orientations of the cone are possible. If C' is positive, the y axis (axis of contraction) will be the axis of the cone. The frontogenetical sector will be contained within the cone, and the frontolytical sector will be all the space outside the cone, as shown in figure 6. If C' is negative, the x axis (axis of dilatation) will be the axis of the cone. The frontolytical sector will be contained within the cone, and the frontogenetical sector will be all the space

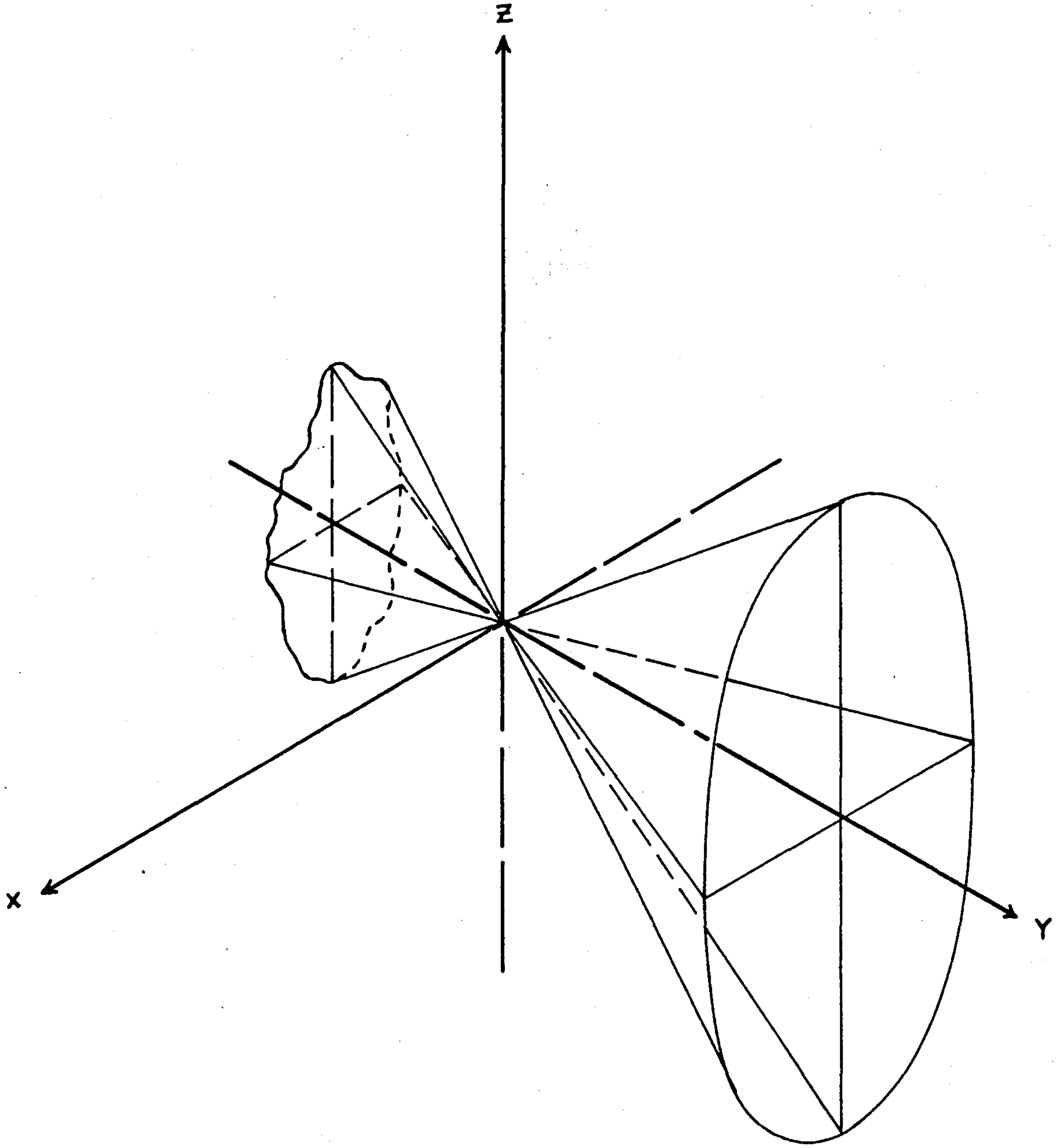

FIGURE 6.

outside the cone, as shown in figure 7.

At this point it is to be emphasized that the cones are symmetric with respect to the axes of dilatation or contraction and not with respect to the axes of outflow or inflow. This holds from the previous developments where it was shown that rotation separates the axes of inflow and outflow from the axes of contraction and dilatation, respectively, while rotation has no effect on the orientation of the frontogenetical and frontolytical sectors.

If divergence predominates over the deformation, there will be no sectors, and no frontogenesis (all frontolysis) will occur. However, if convergence (negative divergence) predominates over the deformation, again, there will be no sectors, but frontogenesis will occur everywhere within the region being considered if the region is linear and if the space rate of change of the orientation of the gradient of property is negligible.

For each of the chosen values of A,B,C there are eight possible orientations of the gradient of property, since l,m,n are each squared terms in the frontogenetical function.

If deformation predominates over divergence, or convergence, the orientation of the gradient of property determines whether or not frontogenesis will occur. If the direction of the gradient of property lies (a) within the frontogenetical sector, (b) on the surface for which

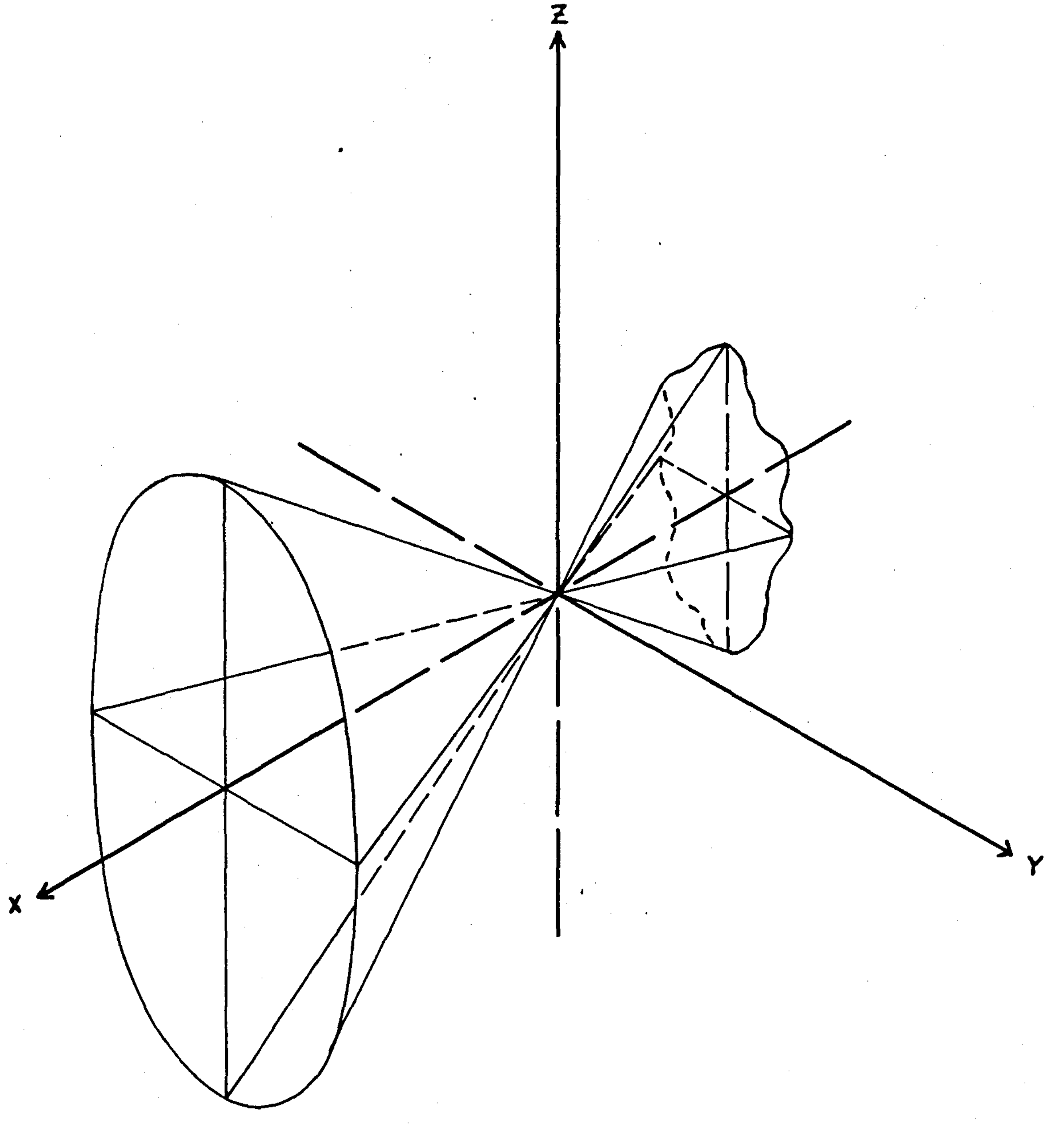

FIGURE 7.

f is equal to zero, or (c) outside the frontogenetical (within the frontolytical) sector, there will take place (a) frontogenesis, (b) no change in the frontogenetical effect, or (c) frontolysis, respectively. These considerations may be applied in any of the eight octants of the coordinate system, since a portion of the sectors is present in each octant, and the direction of the gradient of property may appear in any octant. It is to be noted that these sectors, as defined, are interchanged with those as given by Petterssen (5,6). His sectors are related to the isolines of the conservative property, while the sectors herein are related to the gradient of the property.

If potential temperature is used as the conservative property in the frontogenetical function, the direction of its gradient is nearly vertical. However, under conditions of marked subsidence or convection, the direction of the gradient would depart from the vertical. This point is brought out in the chapter on atmospheric wind systems.

IV. FRONTOGENESIS IN ATMOSPHERIC WIND SYSTEMS.

12. Preliminary Discussion.

The dominant types of atmospheric wind systems observed in the free atmosphere are cols, cyclones, anticyclones, troughs, and wedges. In order that the mathematical developments may apply to these systems, one of the coordinate planes will be assumed nearly horizontal, and then the coordinate axis normal to this plane will be nearly vertical.

The original derivation of the equations of linear fields of motion in section 1 required that the discussion be limited to the immediate vicinity of a point so that the second order and higher terms of the Taylor expansion would be negligible in comparison with the linear terms. In the discussion to follow it must be assumed that the entire region under consideration is a linear velocity field wherein the velocity components are linearly dependent on the distance from the point in question. This point is usually the center of the atmospheric system under consideration.

In section 5, the possible interpretations of divergence were discussed mathematically. There are no pressure fluctuations observed in the atmosphere that represent pure mass divergence or convergence in stable systems of the size being considered. The discussion to follow is limited to stable systems and therefore will assume that divergence is negligible in comparison to deformation and rotation in the shaping of the wind systems observed. By stable systems are meant those systems that retain their identifying features for a reasonable length of time. For these assumptions, the frontogenetical function can be simplified to

$$f = |\nabla S| \left(-A\ell^2 - Bm^2 - Cn^2\right), \tag{12-1}$$

since A+B+C equals zero. Then, in atmospheric systems, choose axes so that A is positive, B is negative, and so that C is positive or negative (equation (6-3)).

The conservative property to be used in this paper is potential temperature. Since the equipotential surfaces are nearly horizontal, the gradient of potential temperature will be nearly vertical. As one approaches the center of an atmospheric system with marked convection or subsidence, the gradient will depart from the vertical by an amount depending primarily on the constant of extension in the vertical. At the very center, however, the direction of the gradient must return to the nearly vertical direction.

13. Frontogenesis in the Vicinity of Cols.

A col is a region of relative calm surrounded by four distinct systems; two diametrically opposite cyclonic, low pressure, systems and two diametrically opposite anticyclonic, high pressure, systems. Petterssen (6) distinguishes three types of cols: (a) without vorticity, in which the relative sizes of the high and low systems with respect to the col are the same, (b) with anti-cyclonic vorticity in which the relative sizes of the low systems predominate over the relative sizes of the high systems, and (c) with cyclonic vorticity in which the relative sizes of the high systems predominate over the relative sizes of the low systems.

In a col the constants of extension predominate over the constants of rotation. The three-dimensional case superimposes either ascending or descending motion on the accepted two-dimensional case as given by Petterssen (5,6) and Stewart (8).

For the xy plane to be nearly horizontal

(13-1) $\quad |A+B| \gg |E| \quad$ and $\quad |A+B| \gg |F|$

or $\quad |C'| \gg |E| \quad$ and $\quad |C'| \gg |F|$.

And for the rotation to be a minimum about the z axis

(13-2) $\quad |D| \ll |B+C| \quad$ and $\quad |D| \ll |A+C|$

or $\quad |D| \ll |A'| \quad$ and $\quad |D| \ll |B'|$.

When these conditions are met, then two types of cols are possible. If C' is positive there is ascending motion, or convection, at the center, while if C' is negative there is descending motion, or subsidence, at the center.

The above restrictions cause the equations for linear fields of motion (2-2) to take the form

(13-3)
$$\begin{aligned} u &= u_0 + A'x \\ v &= v_0 - B'y \\ w &= w_0 \pm C'z \end{aligned} ,$$

if the smaller magnitude terms are considered negligible. This is a special case of the general field. Essentially, these are the equations used to obtain the solutions of section 6 for a pure deformation field. Slight amounts of rotation would cause the coordinate system to become oblique, as shown in figure 4.

It must be recalled from section 10 that the frontogenetical sectors are determined only from the constants of extension. Since, in the case of a col, the constants of extension are large compared to those of rotation, figure 1 superimposed on figure 6 represents frontogenesis in a col with convection, and figure 2 superimposed on figure 7 represents frontogenesis in a col with subsidence.

Figure 1 on 6 shows that the frontogenetical cone is symmetrical with respect to the axis of contraction and that the frontogenetical sector is contained within the cone. It is to be recalled from section 11 that the cone is not affected by rotation even though the axis of inflow does move away from the axis of contraction, and even though the plane of outflow does move away from the plane of dilatation, when rotation is present. Since the direction of the gradient of property is nearly vertical at the center, frontogenesis would not take place in this case with convection. From the preliminary discussion of the preceding section, in the area around the center of the col the direction of the gradient of property may depart sufficiently from the vertical for it to lie within the frontogenetical cone in which case frontogenesis could occur in the direction of the axis of contraction.

Figure 2 on 7 shows that the frontogenetical cone is symmetrical with respect to the axis of dilatation and that the frontogenetical sector is all the space outside the cone. Again, since the direction of the gradient of property is nearly vertical, frontogenesis could take place quite readily in this case. However, around the center of the col the direction of the gradient of property may depart sufficiently from the vertical so that it would lie within the frontolytical sector.

14. Frontogenesis in the Vicinity of Cyclones and Anticyclones.

A cyclone, as used herein, refers to a northern hemisphere extratropical cyclone about which the motion is circular, or nearly circular, in a counterclockwise direction about a center of low pressure. An anticyclone refers to a northern hemisphere region of relatively high pressure about which the motion is circular, or nearly circular, in a clockwise direction.

A cyclone requires that the constants of extension predominate over the constants of rotation in two of the coordinate directions while the reverse must be true in the third direction. The normal cyclone has inflow in the horizontal plane with attendant convection (or outflow) in the vertical direction. The sign convention established in section 6 requires that the yz plane be nearly horizontal and that the x axis be nearly vertical. In the notation being used, these conditions are

(14-1) $\qquad |A'| \gg |D| \quad \& \quad |A'| \gg |E|$

which causes the yz plane to be nearly horizontal, and

(14-2) $\qquad |F| \gg |B'| \quad \& \quad |F| \gg |C'|$

which causes the rotation about the x axis to be a maximum. Further, for cyclonic (counterclockwise) rotation, F must be negative.

The above restrictions cause the equations for linear fields of motion (2-2) to take the form

(14-3)
$$\begin{aligned} u &= u_0 + A'x \\ v &= v_0 - Fz \\ w &= w_0 + Fy \end{aligned} \quad ,$$

if the smaller magnitude terms are considered negligible. This is a special case of the general field.

Figure 5 superimposed on figure 7 shows that the frontogenetical cone is symmetrical with respect to the axis of dilatation (also axis of outflow in this case) and that the frontogenetical sector is all the space outside the cone. Since the direction of the gradient of property is nearly vertical, frontogenesis would not occur in this case. It is observed in the free atmosphere that as one approaches the center of a cyclone, frontal discontinuities are difficult to locate. Again, from the preliminary discussion of section 12, in the area around the center of a cyclone the direction of the gradient of property may depart sufficiently from the vertical for it to lie within the frontogenetical sector, in which case frontogenesis could occur.

An anticyclone requires that the constants of extension and rotation have the same relative magnitudes as they do for a cyclone. However, for an anticyclone the direction of the rotation is reversed. A normal anticyclone has outflow in the horizontal plane with attendant inflow (or subsidence) in the vertical direction. The sign convention of section 6 requires that the xz plane be nearly horizontal and that the y axis be nearly vertical. These conditions are

$$|B'| \gg |D| \quad \& \quad |B'| \gg |F| \tag{14-4}$$

which causes the xz plane to be nearly horizontal, and

$$|E| \gg |A'| \quad \& \quad |E| \gg |C'| \tag{14-5}$$

which causes the rotation about the y axis to be a maximum. Further, for anticyclonic rotation, E must be negative.

The above restrictions cause the equations for linear fields of motion (2-2) to take the form

$$\begin{aligned} u &= u_0 - Ez \\ v &= v_0 - B'y \\ w &= w_0 + Ex \end{aligned} \tag{14-6}$$

if the smaller magnitude terms are considered negligible. This, too, is a special case of the general field.

The frontogenetical cone is symmetrical with respect to the axis of contraction (also axis of inflow in this case), and the frontogenetical sector is contained within the cone. Again, since the direction of the gradient of property is nearly vertical, frontogenesis could occur quite readily in this case. From previous discussions it can be seen that in the area around the center of an anticyclone frontolysis could take place.

15. Frontogenesis in the Vicinity of Troughs and Wedges.

A trough is an elongated area of relatively low pressure extending from the center of a cyclone, while a wedge is an elongated area of relatively high pressure extending from the center of an anticyclone.

The essential difference between troughs and wedges and cyclones and anticyclones is that the region under consideration is no longer the center of the system but is an area outside the influence of the center itself.

The frontogenetical cone has the same orientation it had in the case of the cyclone, since translation has no effect on the frontogenetical function. As in the cyclonic case, the direction of the gradient of property with respect to the frontogenetical sector in a trough would cause frontolysis to occur, assuming the direction to be nearly vertical.

A wedge is analogous to an anticyclone and hence is conducive to frontogenesis.

Until further investigation can be made into the actual magnitudes of the direction cosines of the gradient of property with respect to the linear velocity field constants in troughs and wedges, the following must be inferred from the discussion of cyclones and anticyclones. Since troughs and wedges are being considered as extended areas outside the vicinity of the centers of cyclones and anticyclones, respectively, it is quite possible that the direction of the gradient of property is no longer vertical so that frontogenesis can occur in a trough and frontolysis can occur in a wedge. For example, figure 6 shows that frontogenesis will occur more readily in the direction of the y axis than the z axis due to the elliptical shape of the conical sector.

Petterssen (6), pg. 258, gives an informative description of the relationship between the movements of systems and the movements of fronts. In circular or hyperbolic systems it is necessary for the system to move with nearly the speed of the front for the front to maintain itself. Should the system be stationary, the front would be subject to first frontogenesis, then frontolysis, and then frontogenesis again, etc.

V. CONCLUSIONS.

From the foregoing developments it can be seen that this paper has no more than probed the surface of the topic of Frontogenesis in a Three-Dimensional Linear Velocity Field. In the time available it has not been possible even to parallel all of the topics presented by Petterssen (5,6) for the two-dimensional case. It is the intention of the author to carry forward the investigation commenced herein, not only from a purely theoretical viewpoint, but also along practical lines to see whether or not another useful forecasting tool can be devised.

At its present stage of development this paper should give the reader some idea of the processes that can occur, or are likely to occur, in the formation, persistence, or dissipation of fronts in the free atmosphere. Petterssen (6) gives a complete description of the general frontogenetical regions of the world.

Further investigations should include: (a) research toward finding a more conservative property than potential temperature, (b) the percentage of the frontogenetical sector as related to the total solid angle about the origin of the coordinate system for different values of the constants of extension, (c) the large changes in the direction of the gradient of property in the vicinity of unstable convective cells and the effect on frontogenesis, (d) the effect of variations in the relative magnitudes of A,B,C, (e) applications to actual synoptic situations.

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APPENDIX I.

1. Linear Vector Function of Position.

Let a fixed point in space have the coordinates (Ø,Ø,Ø) in a rectangular coordinate system (x,y,z). And let the velocity at the fixed point be

(A 1-1) $$\underline{V}_o = i\,u_o + j\,v_o + k\,w_o$$

where i,j,k are the unit vectors in the x,y,z directions respectively. Then the vector velocity, $\underline{V}$, in the vicinity of the fixed point can be represented as a vector function of position by an expansion of $\underline{V}$ in a Taylor series:

(A 1-2) $$\underline{V} = \underline{V}_o + \underline{r}\cdot\nabla\underline{V} + \frac{\underline{r}^2}{2!}\cdot\nabla^2\underline{V} + \text{-----}$$

where ∇ (del) has the standard mathematical definition and where the position vector, $\underline{r}$, is

(A 1-3) $$\underline{r} = ix + jy + kz\,.$$

In Cartesian coordinates, neglecting second order and higher terms, $\underline{V}$ can be written as

(A 1-4) $$\begin{aligned} u &= u_o + x\,u_x + y\,u_y + z\,u_z \\ v &= v_o + x\,v_x + y\,v_y + z\,v_z \\ w &= w_o + x\,w_x + y\,w_y + z\,w_z\,. \end{aligned}$$

These equations define a linear velocity field where $\underline{V}$ is a linear vector function of position. This is termed a field of homogeneous deformation.

2. The Equations of Linear Fields of Motion.

Neglecting for the time being the translational terms of equations (A 1-4), the remaining terms can be written in tensor form. Let the tensor that determines the above velocity field be Φ, and let the conjugate of Φ be Φ_c. Then

$$\text{(A 1-5)} \qquad \Phi = \begin{Bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{Bmatrix} = \begin{Bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{Bmatrix}$$

$$\text{(A 1-6)} \qquad \Phi_c = \begin{Bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{Bmatrix} = \begin{Bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Bmatrix} .$$

The equations (A 1-4) involving the components of $\underline{V}$ become

$$\text{(A 1-7)} \qquad \begin{aligned} u &= u_0 + a_{11}x + a_{12}y + a_{13}z \\ v &= v_0 + a_{21}x + a_{22}y + a_{23}z \\ w &= w_0 + a_{31}x + a_{32}y + a_{33}z . \end{aligned}$$

The tensor Φ can be split into a symmetrical part of six terms and an antisymmetrical part of three terms. The symmetrical part represents a velocity of pure extension along the three principal axes, while the antisymmetrical part represents a velocity of rotation. The symmetrical part is $\frac{1}{2}(\Phi + \Phi_c)$, and the antisymmetrical part is $\frac{1}{2}(\Phi - \Phi_c)$.

It can be shown that

$$\text{(A 1-8)} \qquad \Phi = \nabla \underline{V} \qquad \& \qquad \Phi_c = \underline{V} \nabla . \qquad **$$

By direct relationship of the above equations, it is readily verified that

** The above developments of sections 1 and 2 are those of Prandtl and Tietjens (7).

(A 1-9)
$$\tfrac{1}{2}(\Phi+\Phi_c) = \left\{ \begin{matrix} a_{11} & \tfrac{1}{2}(a_{21}+a_{12}) & \tfrac{1}{2}(a_{31}+a_{13}) \\ \tfrac{1}{2}(a_{12}+a_{21}) & a_{22} & \tfrac{1}{2}(a_{32}+a_{23}) \\ \tfrac{1}{2}(a_{13}+a_{31}) & \tfrac{1}{2}(a_{23}+a_{32}) & a_{33} \end{matrix} \right\}$$

(A 1-1Ø)
$$\tfrac{1}{2}(\Phi-\Phi_c) = \left\{ \begin{matrix} 0 & \tfrac{1}{2}(a_{21}-a_{12}) & \tfrac{1}{2}(a_{31}-a_{13}) \\ \tfrac{1}{2}(a_{12}-a_{21}) & 0 & \tfrac{1}{2}(a_{32}-a_{23}) \\ \tfrac{1}{2}(a_{13}-a_{31}) & \tfrac{1}{2}(a_{23}-a_{32}) & 0 \end{matrix} \right\} .$$

Choosing axes so that

(A 1-11)
$$\tfrac{1}{2}(a_{21}+a_{12}) = \tfrac{1}{2}(a_{13}+a_{31}) = \tfrac{1}{2}(a_{23}+a_{32}) = 0$$

then $\tfrac{1}{2}(\Phi+\Phi_c)$ can be written as

(A 1-12)
$$\tfrac{1}{2}(\Phi+\Phi_c) = \left\{ \begin{matrix} a'_{11} & 0 & 0 \\ 0 & a'_{22} & 0 \\ 0 & 0 & a'_{33} \end{matrix} \right\} .$$

By definition, the axes so chosen are the principal axes.

Finally, the equations (A 1-4) involving the components of $\underline{V}$ become

(A 1-13)
$$\begin{aligned} u &= u_0 + (a'_{11}+a'_{22}+a'_{33})x - (a'_{22}+a'_{33})x + \tfrac{1}{2}(a_{12}-a_{21})y + \tfrac{1}{2}(a_{13}-a_{31})z \\ v &= v_0 + (a'_{11}+a'_{22}+a'_{33})y - (a'_{11}+a'_{33})y + \tfrac{1}{2}(a_{21}-a_{12})x + \tfrac{1}{2}(a_{23}-a_{32})z \\ w &= w_0 + (a'_{11}+a'_{22}+a'_{33})z - (a'_{11}+a'_{22})z + \tfrac{1}{2}(a_{31}-a_{13})x + \tfrac{1}{2}(a_{32}-a_{23})y . \end{aligned}$$

For simplification, new symbols are introduced:

TRANSLATION DIVERGENCE DEFORMATION ROTATION

(A 1-14)
$$
\begin{aligned}
u &= u_0 + (A+B+C)x - (B+C)x + Dy + Ez \\
v &= v_0 + (A+B+C)y - (A+C)y - Dx - Fz \\
w &= w_0 + (A+B+C)z - (A+B)z - Ex - Fy
\end{aligned}
$$

where

(A 1-15)
$$
\begin{aligned}
A &= u_x \qquad & D &= -\omega_z \\
B &= v_y \qquad & E &= +\omega_y \\
C &= w_z \qquad & F &= -\omega_x \quad .
\end{aligned}
$$

The equations (A 1-14), involving only nine constants, describe the motion of the most general homogeneous linear velocity field.

APPENDIX II.

Solutions of Linear Differential Equations with Constant Coefficients.

This section applies to the solutions of the differential equations of streamlines when the general field of motion is considered involving all of the constants of equation (A 1-14) except translation. The differential equations are

$$\frac{dx}{Ax+Dy+Ez}=\frac{dy}{By-Dx+Fz}=\frac{dz}{Cz-Ex-Fy} \tag{A 2-1}$$

which are equal to

$$\frac{\lambda\,dx+\mu\,dy+\nu\,dz}{(\lambda A-\mu D-\nu E)x+(\lambda D+\mu B-\nu F)y+(\lambda E+\mu F+\nu C)z} \tag{A 2-2}$$

where λ,μ,ν are constant multipliers.

The expression (A 2-2) will be an exact differential if the numerator is a constant multiple of the differential of the denominator; that is

$$\begin{aligned}\lambda A-\mu D-\nu E&=\rho\lambda\\ \lambda D+\mu B-\nu F&=\rho\mu\\ \lambda E+\mu F+\nu C&=\rho\nu\end{aligned} \tag{A 2-3}$$

These homogeneous linear equations in the multipliers λ,μ,ν may be solved only if the following determinant vanishes:

$$\begin{vmatrix} A-\rho & -D & -E\\ D & B-\rho & -F\\ E & F & C-\rho\end{vmatrix}=0 \tag{A 2-4}$$

This determinant represents a cubic equation that can be written as

(A 2-5) $$\rho^3-(A+B+C)\rho^2+(AB+AC+BC+D^2+E^2+F^2)\rho-(ABC+AF^2+BE^2+CD^2)=0 .$$

For each of the roots ρ_1, ρ_2, ρ_3 of this cubic, the three equations for the multipliers may be solved. Let the resulting denominators of the differential expression (A 2-2) be $\Omega_1, \Omega_2, \Omega_3$ to give

(A 2-6) $$\tfrac{1}{\rho_1}\, d(\ln \Omega_1) = \tfrac{1}{\rho_2}\, d(\ln \Omega_2) = \tfrac{1}{\rho_3}\, d(\ln \Omega_3) .$$

Then the solutions are

(A 2-7) $$\Omega_1^{1/\rho_1} = K_{12}\,\Omega_2^{1/\rho_2} = K_{13}\,\Omega_3^{1/\rho_3} .$$

The developments of this section to this point are by Ford (3).

A direct solution of the cubic in terms of the letter constants is a very laborious algebraic process. For an idea of the possible streamline families, an indirect solution is much easier by assigning arbitrary values to the constants that describe the various types of wind fields in the atmosphere. A nomograph by Lipka (4), Alignment Chart for Solution of Quadratic and Cubic Equations, can be used for the solutions of the cubic where real roots are involved.

It is found that the solutions of the cubic may involve all real roots. The family of streamlines can be found quite readily. The other possibility is that the solution of the cubic involves one real root and two complex conjugate roots. In this case the procedure as outlined below must be followed.

Let ρ_1 be the real root, and let

$$\rho_2 = a + bi$$
$$\rho_3 = a - bi$$

where i in this section refers to the square root of minus one. Then

$$\Omega_1 = \Omega_1(x,y,z) \qquad \text{Real terms only.}$$
$$\Omega_2 = \left[(c+di)x + (e+fi)y + (g+hi)z\right]$$
$$\Omega_3 = \left[(c-di)x + (e-fi)y + (g-hi)z\right]$$

where c,d,e,f,g,h are algebraically complicated functions of a and b.

The solutions of the differential equations will be of the form

(A 2-8) $$\Omega_1^{1/\rho_1} = K_{14}\Omega_2^{1/a+bi} = K_{15}\Omega_3^{1/a-bi} .$$

It can be shown that the third factor of these equations gives the same family of streamlines when used with the first factor as does the second factor, and hence it will not be considered further.

Taking logarithms of both sides of equation (A 2-8),

(A 2-9) $$\frac{a+bi}{\rho_1} \ln \Omega_1 = \ln K_{14} + \ln\left[(cx+ey+gz) + i(dx+fy+hz)\right] .$$

But $$\ln(p+iq) = \ln r + i(\theta + 2n\pi) .$$

Therefore,

(A 2-1Ø)
$$\frac{a+bi}{\beta_1} \ln \Omega_1 = \ln K_{14} + \tfrac{1}{2} \ln\left[(cx+ey+gz)^2+(dx+fy+hz)^2\right] + i \arctan \frac{dx+fy+hz}{cx+ey+gz} \quad .$$

Separating real and imaginary terms gives

(A 2-11)
$$\frac{a}{\beta_1} \ln \Omega_1 = \ln K_{16} + \tfrac{1}{2} \ln\left[(cx+ey+gz)^2+(dx+fy+hz)^2\right]$$
$$\frac{b}{\beta_1} \ln \Omega_1 = \arctan \frac{dx+fy+hz}{cx+ey+gz} \quad .$$

A simultaneous solution of these last equations gives the family of streamlines when complex quantities are involved.

APPENDIX III.

Frontogenesis in Linear Fields of Motion.

This section is intended to furnish the intermediate steps between equations (9-2) and (1∅-2). Equation (9-2) is

(A 3-1) $$f = -\nabla S \cdot (\nabla S \cdot \nabla)\underline{V} / |\nabla S|$$

where ∇S is the gradient of any conservative property. It can be written as

(A 3-2) $$\nabla S = |\nabla S|(\ell i + m j + n K)$$

where l,m,n are the direction cosines of the vector representing the direction of the gradient of the property.

The equations for linear fields of motion (2-2) can be rewritten as follows:

(A 3-3) $$\underline{V} = (u_0 + Ax + Dy + Ez)i + (v_0 + By - Dx + Fz)j + (w_0 + Cz - Ex - Fy)K \; .$$

Substituting this equation into equation (A 3-1), the following steps take place:

$$\nabla S \cdot \nabla = |\nabla S|\left(\ell \tfrac{\partial}{\partial x} + m \tfrac{\partial}{\partial y} + n \tfrac{\partial}{\partial z}\right)$$

$$\begin{aligned}(\nabla S \cdot \nabla)\underline{V} = |\nabla S|\Big[&(\ell u_x + m u_y + n u_z)i \\ &+ (\ell v_x + m v_y + n v_z)j \\ &+ (\ell w_x + m w_y + n w_z)K\Big]\end{aligned}$$

$$\begin{aligned}f = -|\nabla S|\Big[&(\ell^2 u_x + \ell m u_y + \ell n u_z) \\ &+ (\ell m v_x + m^2 v_y + mn v_z) \\ &+ (\ell n w_x + mn w_y + n^2 w_z)\Big] \; .\end{aligned}$$

Replacing the partial derivatives by their letter constants of equations (A 1-15), the above can be rewritten as

(A 3-4) $$f = -|\nabla S| \Big[(\ell^2 A + \ell m D + \ell n E) + (-\ell m D + m^2 B + mn F) + (-\ell n E - mn F + n^2 C) \Big]$$

or,

(A 3-5) $$f = -|\nabla S| \left[\ell^2 A + m^2 B + n^2 C \right]$$

or,

(A 3-6) $$f = |\nabla S| \Big[(B+C)\ell^2 + (A+C) m^2 + (A+B) n^2 - (A+B+C) \Big].$$